Revisiting the Asymptotic Optimality of RRT*

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***Abstract*— RRT* is one of the most widely used sampling-based algorithms for asymptotically-optimal motion planning. This algorithm laid the foundations for optimality in motion planning as a whole, and inspired the development of numerous new algorithms in the field, many of which build upon RRT* itself. In this paper, we first identify a logical gap in the optimality proof of RRT*, which was developed in Karaman and Frazzoli (2011). Then, we present an alternative and mathematically-rigorous proof for asymptotic optimality. Our proof suggests that the connection radius used by RRT* should be increased from $\gamma \left(\frac{\log n}{n}\right)^{1/d}$ to $\gamma' \left(\frac{\log n}{n}\right)^{1/(d+1)}$ in order to account for the additional dimension of time that dictates the samples' ordering. Here γ, γ' are constants, and n, d are the number of samples and the dimension of the problem, respectively.**

I. Introduction

For many robot motion-planning applications, feasibility is not enough—we further desire path plans that are of high quality, reflecting a need for robots that can achieve their goals with efficiency, alacrity, and economy of motion. To this end we seek planning algorithms that can be trusted, whatever obstacle environment a robot faces, to produce optimal or near-optimal plans with minimal scenario-specific tuning. The advent of the asymptotically-optimal rapidly-exploring random tree (RRT*) algorithm [1] has ushered in a decade of theoretical and practical successes in the development of optimal sampling-based motion-planning algorithms.

Although proposed in its initial form for the case of minimum-length path planning for robots without dynamic constraints, RRT* has been extended to handle kinodynamic planning problems [2] including robotic systems governed by non-holonomic constraints [3], more expressive costs accounting for robot energy expenditure [4], [5], and even to plan paths that minimize violation of safety rules [6] or that otherwise balance performance considerations with safety constraints [7]. Heuristic modifications to the core algorithm have also been demonstrated that improve practical RRT* implementations [8], [9].

Each of these extensions leverages the simple yet powerful iterative local graph-rewiring technique introduced by RRT* to enable convergence to the optimal solution (as computation budget increases), provided an appropriate choice for the scaling of the rewiring radius as a function of sample count. Moreover, each of these extensions draws upon the original analysis presented in [1] for the fundamental asymptotic scaling of this algorithm parameter; this analysis is therefore core to each of their optimality guarantees.

Contribution. The primary contribution of this paper is an in-depth study of the theoretical analysis underpinning the asymptotic-optimality criterion for the RRT* algorithm. In revisiting this analysis, we identify a logical gap in the original proof and provide an amended proof suggesting a larger radius scaling exponent to ensure asymptotic optimality. The impact of this paper is potentially far-reaching in the large number of works that currently appeal to RRT* optimality to make their theoretical guarantees.

The paper is organized as follows. Section II provides preliminaries and a description of RRT*. In Section III we review the original optimality proof of RRT* and identify a logical gap within it. In Section IV we provide the main contribution of this paper, which is an alternative proof that circumvents this logical gap. We conclude the paper in Section V.

II. Preliminaries

We provide several basic definitions that will be used throughout the paper. Given two points $x, y \in \mathbb{R}^d$, denote by $\|x - y\|$ the standard Euclidean distance. Denote by $\mathcal{B}_r(x)$ the d-dimensional ball of radius $r > 0$ centered at $x \in \mathbb{R}^d$. Define $\mathcal{B}_r(\Gamma) := \bigcup_{x \in \Gamma} \mathcal{B}_r(x)$ for any $\Gamma \subseteq \mathbb{R}^d$. Similarly, given a curve $\sigma : [0,1] \to \mathbb{R}^d$, define $\mathcal{B}_r(\sigma) = \bigcup_{\tau \in [0,1]} \mathcal{B}_r(\sigma(\tau))$. For a subset $D \subset \mathbb{R}^d$, $|D|$ denotes its Lebesgue measure. All logarithms used herein are to base e.

A. Motion planning

Denote by $\mathcal{C}$ the robot's configuration space, and by $\mathcal{F} \subseteq \mathcal{C}$ the free space, i.e., the set of all collision free configurations. We assume that $\mathcal{C}$ is a subset of the Euclidean space. For simplicity, let $\mathcal{C} = [0,1]^d \subset \mathbb{R}^d$ for some fixed $d \geq 2$. Given start and target configurations $s, t \in \mathcal{F}$, the *motion-planning* problem consists of finding a continuous path (curve) $\sigma : [0,1] \to \mathcal{F}$ such that $\sigma(0) = s$ and $\sigma(1) = t$. That is, the robot starts its motion along σ at s, and ends at t, while avoiding collisions. An instance of the problem is defined by $(\mathcal{F}, s, t)$. We consider the standard path length as a measure of quality:

Definition 1. Given a path σ, its *length* (cost), which corresponds to its Hausdorff measure, is represented by

$$c(\sigma) = \sup_{n \in \mathbb{N}_+, 0 = \tau_1 \leq \ldots \leq \tau_n = 1} \sum_{i=2}^{n} \|\sigma(\tau_i) - \sigma(\tau_{i-1})\|.$$

We proceed to describe the notion of *robustness*, which is essential when discussing theoretical properties of sampling-based planners. Given a subset $\Gamma \subset \mathcal{C}$ and two configurations

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$x, y \in \Gamma$, denote by $\Sigma^{\Gamma}_{x,y}$ the set of all continuous paths, whose image is in Γ, that start in x and end in y, i.e., if $\sigma \in \Sigma^{\Gamma}_{x,y}$ then $\sigma : [0,1] \to \Gamma$ and $\sigma(0) = x, \sigma(1) = y$. We mention that the following definition is slightly different than the one used in [1], [10].

Definition 2. Let $(\mathcal{F}, s, t)$ be a motion-planning problem. A path $\sigma \in \Sigma^{\mathcal{F}}_{s,t}$ is *robust* if there exists $\delta > 0$ such that $\mathcal{B}_{\delta}(\sigma) \subset \mathcal{F}$. We also say that $(\mathcal{F}, s, t)$ is *robustly feasible* if there exists such a robust path.

Definition 3. The *robust optimum* is defined as

$$c^* = \inf \left\{ c(\sigma) \middle| \sigma \in \Sigma^{\mathcal{F}}_{s,t} \text{ is robust} \right\}.$$

B. Algorithms

While our main focus in this paper is the RRT* algorithm, we also rely on the properties of the RRT algorithm, which is described first. The following description of the (geometric) RRT algorithm is based on [11] and [1].

Algorithm 1 RRT $(x_{\texttt{init}} := s, x_{\texttt{goal}} := t, n, \eta)$

$V = \{x_{\text{init}}\}$
for $j = 1$ to n **do**
  $x_{\text{rand}} \leftarrow$ SAMPLE-FREE( )
  $x_{\text{near}} \leftarrow$ NEAREST(x_{rand}, V)
  $x_{\text{new}} \leftarrow$ STEER$(x_{\text{near}}, x_{\text{rand}}, \eta)$
  if COLLISION-FREE$(x_{\text{near}}, x_{\text{new}})$ **then**
    $V = V \cup \{x_{\text{new}}\}$
    $E = E \cup \{(x_{\text{near}}, x_{\text{new}})\}$
return $G = (V, E)$

The input for RRT (Algorithm 1) is an initial and goal configurations $x_{\text{init}}, x_{\text{goal}}$, number of iterations n, and a steering parameter $\eta > 0$. RRT constructs a tree $G = (V, E)$ by performing n iterations. In each iteration, a new sample x_{rand} is returned from $\mathcal{F}$ uniformly at random by calling SAMPLE-FREE. Then, the vertex $x_{\text{near}} \in V$ that is nearest (according to $\|\cdot\|$) to x_{rand} is found using NEAREST. A new configuration $x_{\text{new}} \in \mathcal{X}$ is then returned by STEER, such that x_{new} is on the line segment between x_{near} and x_{rand}, and the distance $\|x_{\text{near}} - x_{\text{new}}\|$ is at most η. Finally, COLLISION-FREE$(x_{\text{near}}, x_{\text{new}})$ checks whether the straight-line path from x_{near} to x_{new} is collision free. If so, x_{new} is added as a vertex to G and is connected by an edge from x_{near}.

We proceed to describe RRT* [1] in Algorithm 2. Every RRT* iteration begins with an RRT-style extension. The difference lies in the subsequent lines. First, RRT* attempts to connect the tree to x_{new} from all its neighbors in V within a $\min\{r(|V|), \eta\}$ vicinity (lines 7-15). Notice that the expression $r(|V|)$ determines the radius based on the current number of vertices in V. (The operation NEAR$(x_{\text{new}}, V, \min\{r(|V|), \eta\})$ returns the subset $V \cap \mathcal{B}_{\min\{r(|V|),\eta\}}(x_{\text{new}})$, i.e., the vertices that are within a distance of $\min\{r(|V|), \eta\}$ from x_{new}.) However, it only adds a single edge to x_{new} from the neighbor $x_{\text{min}} \in X_{\text{near}}$ such that COST(x_{new}) is minimized (line 16). In the next step, RRT* attempts to perform rewires (lines 17-21): with the addition of x_{new}, it may be beneficial to reroute the existing path of x_{near} to use x_{new}. RRT* checks whether changing the parent of x_{near} to be x_{new} reduces COST(x_{near}). (PARENT(x_{near}) returns the immediate predecessor of x_{near} in G. COST(x) for $x \in V$ returns the cost of the path leading from x_{init} to x in G.)

Algorithm 2 RRT* $(x_{\texttt{init}} := s, x_{\texttt{goal}} := t, n, r, \eta)$

1: $V = \{x_{\text{init}}\}$
2: **for** $j = 1$ to n **do**
3: $x_{\text{rand}} \leftarrow$ SAMPLE-FREE( )
4: $x_{\text{near}} \leftarrow$ NEAREST(x_{rand}, V)
5: $x_{\text{new}} \leftarrow$ STEER$(x_{\text{near}}, x_{\text{rand}}, \eta)$
6: **if** COLLISION-FREE$(x_{\text{near}}, x_{\text{new}})$ **then**
7: $X_{\text{near}} =$ NEAR$(x_{\text{new}}, V, \min\{r(|V|), \eta\})$
8: $V = V \cup \{x_{\text{new}}\}$
9: $x_{\text{min}} = x_{\text{near}}$
10: $c_{\text{min}} =$ COST$(x_{\text{near}}) + \|x_{\text{new}} - x_{\text{near}}\|$
11: **for** $x_{\text{near}} \in X_{\text{near}}$ **do**
12: **if** COLLISION-FREE$(x_{\text{near}}, x_{\text{new}})$ **then**
13: **if** COST$(x_{\text{near}}) + \|x_{\text{new}} - x_{\text{near}}\| <$ COST(x_{min}) **then**
14: $x_{\text{min}} = x_{\text{near}}$
15: $c_{\text{min}} =$ COST$(x_{\text{near}}) + \|x_{\text{new}} - x_{\text{near}}\|$
16: $E = E \cup \{(x_{\text{min}}, x_{\text{new}})\}$
17: **for** $x_{\text{near}} \in X_{\text{near}}$ **do**
18: **if** COLLISION-FREE$(x_{\text{new}}, x_{\text{near}})$ **then**
19: **if** COST$(x_{\text{new}}) + \|x_{\text{near}} - x_{\text{new}}\| <$ COST(x_{near}) **then**
20: $x_{\text{parent}} =$ PARENT(x_{near})
21: $E = E \cup \{(x_{\text{new}}, x_{\text{near}})\} \setminus \{(x_{\text{parent}}, x_{\text{near}})\}$
22: **return** $G = (V, E)$

Remark 1. As mentioned above, RRT* performs extensions of the tree in a manner similar to RRT. That is, STEER generates x_{new}, which lies on the straight line connecting $x_{\text{near}}, x_{\text{rand}}$, such that $\|x_{\text{new}} - x_{\text{near}}\| \leq \eta$. Note that initially $x_{\text{new}} \neq x_{\text{rand}}$, but once the space is sufficiently covered by G, i.e., when $\mathcal{F} \subset \bigcup_{v \in V} \mathcal{B}_{\eta}(v)$, then in all the following iterations it will hold that $x_{\text{new}} = x_{\text{rand}}$. This property will be important in the analysis of RRT*, as it indicates that x_{new} is uniformly sampled from $\mathcal{F}$. This notion will be formalized below. For now, it is useful to note that given the same sequence of samples, RRT and RRT* will generate two (possibly distinct) graphs that have a common vertex set.

III. Original Optimality Proof

In this section we review the original proof [1] for asymptotic optimality of RRT*, and point out a logical gap. Specifically, Theorem 38 in [1] states that if the connection radius used by RRT* is of the form

$$r^{\text{KF}}(n) = \gamma \left(\frac{\log n}{n} \right)^{1/d}, \tag{1}$$

where $n \in \mathbb{N}_+$, and for some constant $\gamma > 0$, the cost of the solution obtained by RRT* converges to the robust optimum c^* as $n \to \infty$, almost surely.

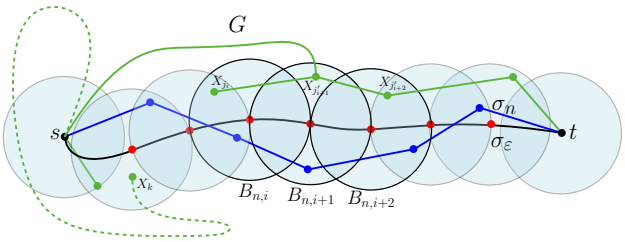

Fig. 1. Illustration of the components in the original proof [1]. (i) The robustly-optimal path σ_ε is drawn as a black curve. (ii) Discs represent the balls $B_{n,1},\ldots,B_{n,M_n}$, whose centers are denoted as red bullets along σ_ε. The path σ_n connecting samples between adjacent balls in an increasing order is illustrated as a blue curve. (iii) A problematic scenario (Section III-B) corresponding to $X^B=\emptyset$ where the RRT* tree G yields a suboptimal solution is depicted in solid green (see Section III-B), where samples that are not drawn in the correct order force the path reaching the target to use detours that increase its cost. (iv) A similar issue, now with $X^B\neq\emptyset$ is depicted where the long path reaching X_k is depicted in dashed green.

A. Review of previous proof

We provide a sketch of the original proof and identify a logical gap. We mention that our definitions of robustness (Definition 2) and robust optimum (Definition 3) are simplified versions of the ones used originally in [1], where the latter are slightly less convenient to work with (especially in correction of the proof which we give in Section IV). We thus adapt the original proof details presented in this section to our setting. We emphasize that the logical gap is unrelated to those definitions, and our argument presented below can be easily remapped to the original formulation.

Recall that the sample set of RRT* consists of n time-labeled configurations. Denote by $\{X_1,\ldots,X_n\}$ the sample set, where indices denote the order in which the samples are drawn. Fix $\varepsilon>0$ and let σ_ε be a robust solution path such that $c(\sigma_\varepsilon)\leq(1+\varepsilon)c^*$. The proof constructs a sequence of M_n identical balls $B_{n,1},\ldots,B_{n,M_n}$ that are centered on some equally-spaced points along σ_ε. The size and spacing of balls is set so that (a) σ_ε is completely covered by them, (b) $\bigcup_{i=1}^{M_n}B_{n,i}\subseteq\mathcal{F}$, and (c) for every $1\leq i\leq M_n, x\in B_{n,i}, x'\in B_{n,i+1}$ it holds that $\|x-x'\|\leq r^{\mathrm{KF}}(n)\leq r^{\mathrm{KF}}(|V|)$. Furthermore, it is shown in [1] that given $x_i\in B_{n,i}$ for every $1\leq i\leq M_n$, the length of the path σ connecting each x_i to the point in the next ball with a straight line converges (as $n\to\infty$) to the length of σ_ε (see Figure 1).

The proof then establishes that if for every $1\leq i<M_n$ there exist $X_{j_i},X_{j_{i+1}}$ such that (i) $X_{j_i}\in B_{n,i}, X_{j_{i+1}}\in B_{n,i+1}$ and (ii) $j_i<j_{i+1}$, then RRT* is asymptotically optimal (see Section G.3 in [1]). The rationale behind these conditions is as follows. Condition (i) makes sure that the optimal path is approximated by samples drawn by RRT*, i.e., for every point along σ_ε there is a sample point in its vicinity. Condition (ii) ensures that RRT* will have the opportunity to add a directed edge from X_{j_i} to $X_{j_{i+1}}$: as $X_{j_{i+1}}$ is sampled after X_{j_i} then RRT* would consider drawing a directed edge from the latter to the former, considering the fact that $X_{j_i}\in\textsc{near}(X_{j_{i+1}},V,\min\{r(n),\eta\})$ (this is formalized in Claim 1 below). Observe that $r(n)$ is used as a conservative lower-bound for $r(|V|)$ throughout [1], as we do as well.

Consequently, the proof deduces that if these conditions are met RRT* is guaranteed to find a solution with cost at most $c^*(1+\varepsilon)$ with probability that converges to 1 as $n\to\infty$. In particular, denote by $X_{j_1},\ldots,X_{j_{M_n}}$ the sequence of samples satisfying the conditions above, and let σ_n be a path that is induced by those M_n samples in the prescribed order. Then the claim is that the solution returned by RRT* is of length $c(\sigma_n)$, if not shorter.

B. A logical gap

We identify an issue with the proof technique described above, and in particular with the conditions (i) and (ii). **We assert that the line of reasoning mentioned above overlooks the fact that the existence of pairwise sequential samples does not directly imply the existence of a whole chain of samples with a proper ordering such that a path in G traces through all the balls in sequence.** That is, the fact that for every $1\leq i<M_n$ (i) there exist $X_{j_i},X_{j_{i+1}}$ such that $X_{j_i}\in B_{n,i}, X_{j_{i+1}}\in B_{n,i+1}$ and (ii) $j_i<j_{i+1}$, does not necessarily mean that (iii) there exists a sequence $j_1\leq j_2\leq\ldots\leq j_{M_n}$ such that $X_{j_i}\in B_{n,i}$ for every $1\leq i<M_n$; (iii) is a sufficient (but not necessary) condition for recovering a path that is at least as good as σ_n.

Consider for instance the case where $X_{j_i}\in B_{n,i}, X_{j_{i+1}}\in B_{n,i+1}, X_{j'_{i+1}}\in B_{n,i+1}, X_{j'_{i+2}}\in B_{n,i+2}$ and $j_i<j_{i+1}, j'_{i+1}<j'_{i+2}$, but $j'_{i+2}<j_i$, where there are two points $X_{j_{i+1}},X_{j'_{i+1}}$ that fall into the same ball $B_{n,i+1}$. We provide a few examples for problematic cases that can arise in this situation. Define $X_1^{j_i-1}=\{X_1,\ldots,X_{j_i-1}\}, B_1^{i-1}=\bigcup_{k=1}^{i-1}B_{n,i-1}$, and let $X^B=X_1^{j_i-1}\cap B_1^{i-1}\cap\mathcal{B}_{r(j_i)^{\mathrm{KF}}}(X_{j_i})$. Namely, X^B contains all the sampled points that were drawn before X_{j_i}, which lie in previous balls along σ_ε, and whose distance from X_{j_i} is at most $r(j_i)^{\mathrm{KF}}$.

If $X^B=\emptyset$, this implies that in iteration j_i of RRT*, X_{j_i} would either be connected via a sample that is entirely outside $\bigcup_{i=1}^{M_n}B_{n,i}$, or via some point in $B_{n,i+1}$ (possibly $X_{j'_{i+1}}$). In both cases however, the path reaching X_{j_i} would not be fully contained in B_1^{i-1}, and thus can be quite long, when compared with the corresponding path segment of σ_ε reaching $B_{n,i}$. Note that in iteration j_{i+1} the addition of sample $X_{j_{i+1}}$ would not resolve this problematic wiring. See Figure 1 (iii).

Now, assume otherwise that $X^B\neq\emptyset$ and let X_k be a sample such that $X_k\in X^B$. Additionally, assume that

$$\textsc{cost}(X_{j'_{i+1}})+\|X_{j_i}-X_{j'_{i+1}}\|<\textsc{cost}(X_k)+\|X_{j_i}-X_k\|.$$

Such a situation can occur due to the stochastic nature of the algorithm, and in particular when there are several free-space corridors through which $X_k, X_{j'_{i+1}}$ may be reached, and it is easier to reach X_k through a long path, than through a short one which goes along σ_ε. Consequently, when X_{j_i} will be drawn in iteration j_i, RRT* would either connect to X_{j_i} through $X_{j'_{i+1}}$ or via another sample that lies outside of X^B. See Figure 1 (iv).

As we show in our proof in the next section, condition (iii) is in fact sufficient to guarantee asymptotic optimality, and we prove that it indeed holds with high probability when we slightly increase the connection radius from Equation (1), as well as slightly modify the constant γ.

IV. Alternative Proof

In order to account for the additional dimension of time, we set the connection radius to be $r(n) = \gamma\left(\frac{\log n}{n}\right)^{\frac{1}{d+1}}$, where γ is a constant that will be determined below. We state our main theorem and provide an overview of the proof. The full proof is presented later on. Note that our result suggests that the exponent should be increased from $1/d$ to $1/(d+1)$, which yields a larger radius overall. Denote by σ_n the path connecting s to t returned by RRT* after n iterations. In case that RRT* fails to find any solution, we denote $c(\sigma_n) = \infty$. Recall that η is the steering parameter of RRT*.

Theorem 1. *Suppose that $(\mathcal{F}, s, t)$ is robustly feasible, fix $\eta > 0, \varepsilon \in (0,1), \theta \in (0, 1/4), \mu > 0$, and define the radius of* RRT* *to be*

$$r(n) = \gamma\left(\frac{\log n}{n}\right)^{\frac{1}{d+1}}, \tag{2}$$

$$\gamma > (2+\theta)\left(\frac{(1+\varepsilon/4)c^*}{(d+1)\theta(1-\mu)} \cdot \frac{|\mathcal{F}|}{\zeta_d}\right)^{\frac{1}{d+1}}, \tag{3}$$

where ζ_d is the volume of a unit d-dimensional hypersphere, c^* is the robust optimum. Then

$$\lim_{n\to\infty} \Pr[c(\sigma_n) \le (1+\varepsilon)c^*] = 1.$$

Our proof of Theorem 1 proceeds similarly to the proof of the asymptotic optimality of FMT* [10] (which is in turn based on [1]), but with additional complications due to the time dimension and the coupling with the RRT algorithm. We proceed to describe the main ingredients of the proof.

Fix the parameters $\theta \in (0, 1/4), \mu > 0, \eta > 0$. Also define the constants $\varepsilon > 0$, $\alpha \in (0, \theta\varepsilon/16), \beta \in (0, \theta\varepsilon/16)$, which will be used in the proof. Due to the fact that $(\mathcal{F}, s, t)$ is robustly feasible, there exists a robust path $\sigma_\varepsilon \in \Sigma^{\mathcal{F}}_{s,t}$ and $\delta > 0$ such that $c(\sigma_\varepsilon) \le (1+\varepsilon/4)c^*$ and $\mathcal{B}_\delta(\sigma_\varepsilon) \subset \mathcal{F}$. We will show that the RRT* graph G contains a path that is in the vicinity of σ_ε, which implies that the solution returned by RRT* is of cost at most $(1+\varepsilon)c^*$ (which is slightly larger than $(1+\varepsilon/4)c^*$ due to the fact that this is still an approximation of the path σ_ε).

The first part of the proof deals with the technicality involved with the samples produced by the algorithm. Denote by $V = \{X_1 \ldots, X_n\}$ the sequence of vertices produced by RRT*, where X_j is equal to x_{new} generated in iteration j. Due to the fact that RRT* (and RRT) perform steering (line 5), samples are not distributed in a uniform manner, as x_{rand} is not necessarily identical to x_{new} (see Remark 1). However, we do show that most of the vertices in V that are in the vicinity of σ_ε are distributed uniformly at random, with probability approaching 1 (see Lemma 1). This event is denoted by $\mathfrak{E}^1$ (see Definition 4).

Next, we proceed in a manner similar to other proofs of asymptotic optimality (see, [1], [10], [12]), by defining a sequence of points $x_1, \ldots, x_{M_n}$ along the path σ_ε and specifying a sequence of balls $B_{n,1}, \ldots, B_{n,M_n}$ that are centered on those points respectively, and whose radius is proportional to $r(n)$. More formally, define $M_n = \left\lceil c(\sigma_\varepsilon) \cdot \left(\frac{r(n)}{2+\theta}\right)^{-1} \right\rceil$, and let $x_1, \ldots, x_{M_n}$ be a sequence of points along σ_ε such that $\|x_i - x_{i-1}\| \le \frac{\theta r(n)}{2+\theta}$, $x_1 = s, x_{M_n} = t$. For every $1 \le i \le M_n$ define $B_{n,i} := \mathcal{B}_{\frac{r(n)}{2+\theta}}(x_i)$.

As suggested in Section III, we need to reason both about the existence of samples inside those balls, and the order of those samples. We assign to every ball $B_{n,i}$ a specific time window T_i, corresponding to allowed timestamps of samples, and partition the sample set $V = \{X_1, \ldots, X_n\}$ into the subsets $V_0, V_1, \ldots, V_{M_n}$, where $X_j \in V_i$ if $j \in T_i$. In particular, T_0 consists of the first n' indices, where $n' = \mu n$, and every T_i, where $i > 1$ consists of $(n-n')/M_n$ indices, and $\mu \in (0,1)$ is a constant:

$$T_{i\neq 0} = \left\{ n' + (i-1) \cdot \left\lfloor \tfrac{n-n'}{M_n} \right\rfloor + 1, \ldots, n' + i \cdot \left\lfloor \tfrac{n-n'}{M_n} \right\rfloor \right\}.$$

We show that the event $\mathfrak{E}^2$ (Definition 5) indicating that every $B_{n,i}$ contains a vertex from V_i occurs with probability approaching 1 as well (Lemma 2). The motivation for this event is the following claim, which indicates that edges between points in consecutive balls are added if deemed beneficial.

Claim 1. *There exists $n \in \mathbb{N}_+$ large enough such that the following holds with respect to $G_{j_{i+1}} = (V_{j_{i+1}}, E_{j_{i+1}})$: Suppose that there exist $X_{j_i} \in V_i \cap B_{n,i}, X_{j_{i+1}} \in V_{i+1} \cap B_{n,i+1}$ and denote by $G_{j_{i+1}}$ the* RRT* *graph at the end of iteration j_{i+1}. Then in G_{j_i+1} it follows that* COST$(X_{j_{i+1}}) \le$ COST$(X_{j_i}) + \|X_{j_i} - X_{j_{i+1}}\|$.

Proof. Recall that $B_{n,i} = \mathcal{B}_{\frac{r(n)}{2+\theta}}(x_i)$ and $\|x_i - x_{i+1}\| \le \frac{\theta r(n)}{2+\theta}$. For any $x \in B_{n,i}, x' \in B_{n,i+1}$ it follows that

$$\|x - x'\| \le \tfrac{r(n)}{2+\theta} + \tfrac{\theta r(n)}{2+\theta} + \tfrac{r(n)}{2+\theta} = r(n).$$

This implies that $X_{j_i} \in X_{\text{near}} =$ NEAR$(X_{j_i}, V_{j_{i+1}}, r(n))$, which will cause the execution of the test COLLISION-FREE$(X_{j_i}, X_{j_{i+1}})$ (line 12 of RRT*). The latter will be evaluated to be true since $\mathcal{B}_\delta(\sigma_\varepsilon) \subseteq \mathcal{F}$ and $r(n) \ll \delta$ (for n large enough). Thus, in line 13 the edge $(X_{j_i}, X_{j_{i+1}})$ will be added to the graph, unless there is a lower-cost alternative for connection. □

Thus, $\mathfrak{E}^2$ guarantees that the RRT* tree G contains a path σ'_n connecting s to t that follows σ_ε closely. In order to ensure that $c(\sigma'_n) \le (1+\varepsilon)c^*$ we need one more step, since σ'_n could stay close to σ_ε but zig-zag around it, resulting in a high-cost solution. For the previously defined α, β, define for every $1 \le i \le M_n$ the ball $\beta^\beta_{n,i} := \mathcal{B}_{\frac{\beta r(n)}{2+\theta}}(x_i)$. The event $\mathfrak{E}^3$ (Definition 6) indicates that a fraction of at most α of the smaller balls $B^\beta_{n,i}$ does not contain samples from V_i. We show that $\mathfrak{E}^3$ occurs with probability approaching 1 (Lemma 3). We then proceed to show that if $\mathfrak{E}^2, \mathfrak{E}^3$ occur

simultaneously then RRT* is guaranteed to return a solution with cost at most $(1+\varepsilon)c^*$ (Lemma 4).

A. Proof of Theorem 1

We start with a formal definition of $\mathfrak{E}^1$:

Definition 4. For every $1 \le j \le n$ denote by $x^j_{\text{rand}}, x^j_{\text{new}}$ the random and new samples of RRT* in iteration j (line 3 and line 5 in Algorithm 2, respectively). Define $n' := \mu n$ and

$$\mathfrak{E}^1_n := \{\forall 1 \le i \le M_n, n' \le j \le n : \text{if } x^j_{\text{rand}} \in B_{n,i} \text{ then } x^j_{\text{rand}} = x^j_{\text{new}}\}.$$

That is, $\mathfrak{E}^1_n$ is the event that all $x^j_{\text{rand}} \in B_{n,i}$ for j between n' and n satisfy $x^j_{\text{rand}} = x^j_{\text{new}}$.

Remark 2. We wish to stress that the following lemma, which lower bounds the probability of $\mathfrak{E}^1_n$, is a key ingredient in our proof. As we shall see below, this would allow us to treat some of the vertices added by RRT* as uniformly sampled, which is not true for all samples, as some are perturbed by the STEER operation. We mention that this issue was not addressed in the original proof in [1], where the RRT* nodes were assumed (incorrectly) to be uniformly distributed. Furthermore, setting the steering step $\eta = \infty$ does not resolve this issue.

Lemma 1. *There exist two constants $a, b > 0$ such that $\Pr[\mathfrak{E}^1_n] \ge 1 - a \cdot e^{-bn}$.*

Proof. A similar proof appears in [12, Claim 6], albeit for a different type of sampling scheme and in the context of an RRG analysis. The main challenge here is to show that while it is not true that all the new samples x_{new} are distributed uniformly randomly (due to lines 4,5 in Algorithm 2), most of them are. Define $\kappa := \min\{\eta, \delta\}/10$ and set $z_1, \ldots, z_\ell$ to be a sequence of points placed along σ_ε, such that $\ell = c(\sigma_\varepsilon)/\kappa$, and $\|z_k - z_{k+1}\| \le \kappa$. Observe that for n large enough it holds that $\bigcup_{i=1}^{M_n} B_{n,i} \subset \bigcup_{k=1}^{\ell} \mathcal{B}_\kappa(z_k)$.

Denote by $V^{\text{RRT}}_{n'}$ the vertex set of RRT after n' iterations. Theorem 1 in [13] states that there exist constants $a, c > 0$ such that the probability that for every $1 \le k \le \ell$ it holds that $V^{\text{RRT}}_{n'} \cap \mathcal{B}_\kappa(z_k) \neq \emptyset$ is at least $a \cdot e^{-cn'} = a \cdot e^{-bn}$, where $b := c\mu$. Notice that this theorem requires η to be fixed (i.e., independent of n) and strictly positive.

Denote the latter event to be $\mathfrak{E}'^1_n$. Next, we show that $\mathfrak{E}'^1_n$ implies $\mathfrak{E}^1_n$. First, observe that $V^{\text{RRT}}_{n'} = V^{\text{RRT}^*}_{n'}$, where the latter is the vertex set of RRT* after n' iterations, and assume that $\mathfrak{E}'^1_n$ holds . Fix an iteration $n' < j < n$ and some $1 \le k \le \ell$. Due to the fact that $\eta > 0$ is fixed, by the proof of Lemma 1 in [13] it follows that if $x^j_{\text{rand}} \in \mathcal{B}_\kappa(z_j)$ then $x^j_{\text{near}} \in \mathcal{B}_{5\kappa}(z_j)$, and consequently

$$\|x^j_{\text{rand}} - x^j_{\text{near}}\| = \|x^j_{\text{rand}} - z_j + z_j - x^j_{\text{near}}\| \le \|x^j_{\text{rand}} - z_j\| + \|z_j - x^j_{\text{near}}\| \le \kappa + 5\kappa \le \eta.$$

This implies that $x^j_{\text{new}} = x^j_{\text{rand}}$. Additionally, observe that due to the fact that the straight-line path from x^j_{near} to x^j_{rand} is contained in $\mathcal{B}_\kappa(z_j)$, where $\kappa < \delta/5$, it is also collision free. Thus, at the end of iteration j, x_{rand} will be added to the RRT* graph as a vertex. □

We will prove that the following event $\mathfrak{E}^2$ holds with probability approaching 1 by conditioning on $\mathfrak{E}^1$.

Definition 5. $\mathfrak{E}^2_n$ represents the event that every $B_{n,i}$ contains at least one vertex from V_i. That is,

$$\mathfrak{E}^2_n := \{\forall 1 \le i \le M_n, V_i \cap B_{n,i} \neq \emptyset\}.$$

Lemma 2. $\lim_{n\to\infty} \Pr[\mathfrak{E}^2_n] = 1$.

Proof. Observe that

$$\Pr[\mathfrak{E}^2_n] = \Pr[\mathfrak{E}^2_n|\mathfrak{E}^1_n] \cdot \Pr[\mathfrak{E}^1_n] + \Pr[\mathfrak{E}^2_n|\overline{\mathfrak{E}^1_n}] \cdot \Pr[\overline{\mathfrak{E}^1_n}] \ge \Pr[\mathfrak{E}^2_n|\mathfrak{E}^1_n] \cdot \Pr[\mathfrak{E}^1_n].$$

We shall lower-bound the expression $\Pr[\mathfrak{E}^2_n|\mathfrak{E}^1_n]$. By definition of $\mathfrak{E}^1_n$, for every $n' < j \le n$, and i such that $j \in T_i$, if $x^j_{\text{rand}} \in B_{n,i}$, then $x^j_{\text{new}} = x^j_{\text{rand}}$ is a valid vertex of the RRT* graph. Thus, by conditioning on $\mathfrak{E}^1_n$ we can treat $V \setminus V_0$ as uniform random samples from $\mathcal{F}$. This will come in handy in bounding the probability of $\mathfrak{E}^2$:

$$\begin{aligned}
\Pr[\overline{\mathfrak{E}^2_n}|\mathfrak{E}^1_n] &= \Pr[\exists 1 \le i \le M_n, V_i \cap B_{n,i} = \emptyset] \\
&\le \sum_{i=1}^{M_n} \Pr[V_i \cap B_{n,i} = \emptyset] = \sum_{i=1}^{M_n} \left(1 - \frac{|B_{n,i}|}{|\mathcal{F}|}\right)^{|T_i|} && (4)\\
&\le M_n \left(1 - \frac{\zeta_d \left(\frac{r(n)}{2+\theta}\right)^d}{|\mathcal{F}|}\right)^{(n-n')/M_n} \\
&\le M_n \exp\left\{-\frac{n-n'}{M_n} \cdot \frac{\zeta_d}{|\mathcal{F}|} \cdot \frac{r(n)^d}{(2+\theta)^d}\right\} && (5)\\
&\le M_n \exp\left\{-\frac{nr(n)\theta(1-\mu)}{c(\sigma_\varepsilon)(2+\theta)} \cdot \frac{\zeta_d}{|\mathcal{F}|} \cdot \frac{r(n)^d}{(2+\theta)^d}\right\} \\
&= M_n \exp\left\{-\frac{\theta\zeta_d(1-\mu)}{c(\sigma_\varepsilon)(2+\theta)^{d+1}|\mathcal{F}|} \cdot n \cdot r(n)^{d+1}\right\} \\
&=: M_n \exp\left\{-\xi \cdot n \cdot \gamma^{d+1} \frac{\log n}{n}\right\} && (6)\\
&= \left\lceil c(\sigma_\varepsilon) \cdot \left(\tfrac{r(n)}{2+\theta}\right)^{-1} \right\rceil \exp\left\{-\xi\gamma^{d+1} \log n\right\} \\
&< \left(c(\sigma_\varepsilon) \cdot \left(\tfrac{r(n)}{2+\theta}\right)^{-1} + 1\right) \exp\left\{-\xi\gamma^{d+1} \log n\right\} \\
&= \tfrac{c(\sigma_\varepsilon)(2+\theta)}{\theta\gamma} (\log n)^{-1/(d+1)} n^{1/(d+1)-\xi\gamma^{d+1}} \\
&\quad + \exp\left\{-\xi\gamma^{d+1} \log n\right\}, && (7)
\end{aligned}$$

where (4) is due to the union bound and the fact that V_i is uniformly sampled at random from $\mathcal{F}$, (5) is due to the inequality $1 - x \le e^{-x}$ for $x \in (0,1)$ which applies here for n large enough, and (6) defines $\xi := \frac{\theta\zeta_d(1-\mu)}{c(\sigma_\varepsilon)(2+\theta)^{d+1}|\mathcal{F}|}$. If $(d+1)^{-1} - \xi\gamma^{d+1} \le 0$ then the final expression tends to 0. Indeed,

$$\frac{1}{d+1} - \frac{\theta\zeta_d(1-\mu)}{c(\sigma_\varepsilon)(2+\theta)^{d+1}|\mathcal{F}|} \cdot \gamma^{d+1} \le 0 \iff (2+\theta)\left(\tfrac{c(\sigma_\varepsilon)|\mathcal{F}|}{(d+1)\theta\zeta_d(1-\mu)}\right)^{\frac{1}{d+1}} \le (2+\theta)\left(\tfrac{c^*(1+\varepsilon/4)|\mathcal{F}|}{(d+1)\theta\zeta_d(1-\mu)}\right)^{\frac{1}{d+1}} \le \gamma.$$

It remains to show that $\lim_{n\to\infty} \Pr[\mathfrak{E}^2_n|\mathfrak{E}^1_n] \cdot \Pr[\mathfrak{E}^1_n] = 1$:

$$\Pr[\mathfrak{E}^2_n|\mathfrak{E}^1_n] \cdot \Pr[\mathfrak{E}^1_n] = (1 - \Pr[\overline{\mathfrak{E}^2_n}|\mathfrak{E}^1_n])(1 - \Pr[\overline{\mathfrak{E}^1_n}])$$

$$= 1 + \Pr[\overline{\mathfrak{E}_n^2}|\mathfrak{E}_n^1] \cdot \Pr[\overline{\mathfrak{E}_n^1}] - \Pr[\overline{\mathfrak{E}_n^2}|\mathfrak{E}_n^1] - \Pr[\overline{\mathfrak{E}_n^1}]$$
$$> 1 - \Pr[\overline{\mathfrak{E}_n^2}|\mathfrak{E}_n^1] - \Pr[\overline{\mathfrak{E}_n^1}],$$

where the final expression converges to 1, according to Equation 7 and Lemma 1. □

Next we consider the existence of samples in a collection of smaller balls.

Definition 6. Define

$$K_n^\beta := |\{i \in \{1, \ldots, M_n\} : B_i^\beta \cap V_i = \emptyset\}|.$$

$\mathfrak{E}_n^3$ is the event that a fraction of at most α of the smaller balls $B_{n,i}^\beta$ do not contain any samples from V_i, i.e.,

$$\mathfrak{E}_n^3 := \{K_n^\beta \leq \alpha M_n\}.$$

Lemma 3. $\lim_{n\to\infty} \Pr[\mathfrak{E}_n^3] = 1$.

Proof. Similarly to Lemma 2, it is sufficient to show that $\lim_{n\to\infty} \Pr[\overline{\mathfrak{E}_n^3}|\mathfrak{E}_n^1] = 0$. We shall upper bound the probability that $K_n^\beta > \alpha M_n$ assuming that $\mathfrak{E}_n^1$ holds. To this end, we compute the expectation of K_n^β and apply Markov's inequality.

For every $1 \leq i \leq M_n$, denote by I_i the indicator variable for the event that $B_{n,i}^\beta \cap V_i = \emptyset$. Observe that $K_n^\beta = \sum_{i=1}^{M_n} I_i$. For n large enough we have that

$$E[I_i] = \Pr[I_i = 1] = \left(1 - \frac{|B_{n,i}^\beta|}{|\mathcal{F}|}\right)^{|T_i|}$$
$$\leq \left(1 - \frac{\beta^d \zeta_d \left(\frac{r(n)}{2+\theta}\right)^d}{|\mathcal{F}|}\right)^{n(1-\mu)/M_n}$$
$$\leq \exp\left\{-\frac{\beta^d \theta \zeta_d (1-\mu)}{c(\sigma_\varepsilon)(2+\theta)^{d+1}|\mathcal{F}|} \cdot n \cdot r(n)^{d+1}\right\}$$
$$\leq \exp\left\{-\frac{\beta^d \theta \zeta_d (1-\mu)}{c(\sigma_\varepsilon)(2+\theta)^{d+1}|\mathcal{F}|} \cdot \gamma^{d+1} \cdot \log n\right\}$$
$$= \exp\left\{-\frac{\beta^d}{d+1} \log n\right\} = n^{-\beta^d/(d+1)}.$$

Thus,

$$E[K_n^\beta] = \sum_{i=1}^{M_n} E[I_i] \leq M_n n^{-\beta^d/(d+1)}.$$

By Markov's inequality, it follows that

$$\Pr[K_n^\beta > \alpha M_n] \leq \frac{E[K_n^\beta]}{\alpha M_n} \leq \frac{M_n n^{-\beta^d/(d+1)}}{\alpha M_n} = \frac{n^{-\beta^d/(d+1)}}{\alpha}. \quad (8)$$

As α is fixed, the last expression tends to 0 as n tends to ∞. While the upper bound obtained in (8) is sufficient for our purpose, we mention that a tighter bound can be derived by using a slightly more complex Poissonization argument similar to that used in [10]. □

We are ready for the final argument. We show that if $\mathfrak{E}^2, \mathfrak{E}^3$ occur simultaneously, then two conclusions must follow.

Lemma 4. *For n large enough, if the events $\mathfrak{E}_n^2, \mathfrak{E}_n^3$ occur, then $c(\sigma_n) \leq (1+\varepsilon)c^*$.*

Proof. As $\mathfrak{E}_n^2 \wedge \mathfrak{E}_n^3$ we may define the sequence of vertices $X_{j_1}, \ldots, X_{j_{M_n}} \in V$, such that $X_{j_1} = s, X_{j_{M_n}} = t$, and for every $1 < i < M_n$, $X_{j_i} \in V_i \cap B_{n,i}^\beta$ if $V_i \cap B_{n,i}^\beta \neq \emptyset$, and $X_{j_i} \in V_i \cap B_{n,i}$ otherwise.

Denote by σ'_n the path induced by concatenating those points, and notice that it is collision free by definition of $B_{n,i}$ and σ_ε. Next, we claim that the cost of the path σ_n obtained by RRT* is upper-bounded by the cost of σ'_n, which is equal to $\sum_{i=2}^{M_n} \|X_{j_i} - X_{j_{i-1}}\|$. Consider iteration j_i of RRT*, for $1 < i \leq M_n$ and observe that (i) $x_{\text{new}}^{j_i} = X_{j_i}$, (ii) $X_{j_{i-1}} \in X_{\text{near}}^{j_i}$. By Claim 1, it follows that $\text{COST}(X_{j_i}) \leq \sum_{k=2}^{i} \|X_{j(k)} - X_{j(k-1)}\|$, as desired. Thus, $c(\sigma_n) \leq c(\sigma'_n)$.

We proceed to bound $c(\sigma'_n)$. Observe that for any $1 < i \leq M_n$ it holds that $\|X_{j_i} - X_{j_{i-1}}\|$ is at most

$$\begin{cases} \frac{\theta r(n)}{2+\theta} + \frac{\beta r(n)}{2+\theta} + \frac{\beta r(n)}{2+\theta}, & \text{if } X_{j_{i-1}} \in B_{n,i-1}^\beta \text{ AND } X_{j_i} \in B_{n,i}^\beta \\ \frac{\theta r(n)}{2+\theta} + \frac{\beta r(n)}{2+\theta} + \frac{r(n)}{2+\theta}, & \text{if } X_{j_{i-1}} \in B_{n,i-1}^\beta \text{ XOR } X_{j_i} \in B_{n,i}^\beta \\ \frac{\theta r(n)}{2+\theta} + \frac{r(n)}{2+\theta} + \frac{r(n)}{2+\theta}, & \text{otherwise} \end{cases}.$$

Thus,

$$c(\sigma'_n) \leq \sum_{i=2}^{M_n} \|X_{j_i} - X_{j_{i-1}}\|$$
$$\leq (M_n - 1)\frac{\theta r(n)}{2+\theta} + \lceil (1-\alpha)(M_n - 1) \rceil \frac{2\beta r(n)}{2+\theta}$$
$$+ \lfloor \alpha(M_n - 1) \rfloor \frac{2r(n)}{2+\theta} \leq (M_n - 1) r(n) \frac{\theta + 2\beta + 2\alpha}{2+\theta}$$
$$\leq \frac{c(\sigma_\varepsilon)(2+\theta)}{\theta r(n)} r(n) \frac{\theta + 2\beta + 2\alpha}{2+\theta} \leq \left(1 + \tfrac{\varepsilon}{4}\right) c^* \cdot \frac{\theta + 2\beta + 2\alpha}{\theta}$$
$$< \left(1 + \tfrac{\varepsilon}{4}\right) c^* \frac{\theta + \frac{2\theta\varepsilon}{16} + \frac{2\theta\varepsilon}{16}}{\theta} = \left(1 + \tfrac{\varepsilon}{4}\right)^2 c^*$$
$$= \left(1 + \tfrac{\varepsilon}{2} + \tfrac{\varepsilon^2}{16}\right) c^* < \left(1 + \tfrac{\varepsilon}{2} + \tfrac{\varepsilon}{16}\right) c^* < (1+\varepsilon)c^*. \quad \square$$

It remains to show that $\mathfrak{E}^2 \wedge \mathfrak{E}^3$ occurs with probability approaching 1. The next step concludes the proof:

$$\lim_{n\to\infty} \Pr[\mathfrak{E}^2 \wedge \mathfrak{E}^3] = 1 - \lim_{n\to\infty} \Pr[\overline{\mathfrak{E}^2} \vee \overline{\mathfrak{E}^3}]$$
$$\geq 1 - \lim_{n\to\infty} \left(\Pr[\overline{\mathfrak{E}^2}] + \Pr[\overline{\mathfrak{E}^3}]\right) = 1.$$

V. Conclusion

In this paper we revisited the original asymptotic-optimality proof of RRT* in [1], and discussed an apparent logical gap within it. We then introduced an alternative proof that amends this logical gap. Our new proof suggests that the connection radius of RRT* should be slightly larger (Theorem 1) than the original bound on the radius that was developed in [1]. We leave the question of whether our bound is tight, i.e., whether the exponent of $\frac{1}{d+1}$ in Equation (2) can be lowered to $\frac{1}{d}$, to future research. The practical successes of the algorithm and its extensions, using the exponent $\frac{1}{d}$, provide some evidence that this might be the case.

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